2 Indices And Logarithms

M11CH2

2.1. INDICES

Index Notation

Let a be a non-zero real number and n be a positive integer. Then $a \times a \times a \times \ldots \times a$ (n factors) is briefly written as a^n. a^n reads as 'nth power of a'. In a^n, the real number a is called the **base** and the positive integer n is called the **index (**also called **power** or **exponent)**. Thus, a^n is the product of n factors, each equal to a. Hence, $\boldsymbol{a}^n$ can be written as:

$$a^n = a \times a \times a \times \ldots \times a \ (n \text{ factors})$$

In general, the index (n) can be positive, negative or a fraction.

A way for writing number(s) in a^n form is called **index notation** or **index form** or **exponential form**. An index (the small ^{n}) tells us how many times the base number (a) is multiply together.

For example:

$1000 = 10 \times 10 \times 10 = 10^3$.

Thus, 10^3 is called the index form or exponential form of 1000.

> **Note:** Index form of a number can also be written in a product of powers of its prime factors as: $2^x \times 3^y \times 5^y \times \ldots$.

Example 1: *Express each of the following numbers in index form (exponential form):*

(i) 128 (ii) 10000 (iii) 3125

Solution:

(i) $128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$

(ii) $10000 = 10 \times 10 \times 10 \times 10 = 10^4$

(iii) $3125 = (5 \times 5 \times 5 \times 5 \times 5) = 5^5$

Example 2: *Express each of the following numbers as product of powers of prime numbers:*

(i) 432 (ii) 588 (iii) 2700

Solution:

(i) $432 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$ (By factorisation)
$= 2^2 \times 3^4$ (Index form or notion)

(ii) $588 = 2 \times 2 \times 3 \times 7 \times 7$ (By factorisation)
$= 2^2 \times 3^1 \times 7^2$(Index form)

(iii) $2700 = 2 \times 2 \times 5 \times 5 \times 3 \times 3 \times 3$ (By factorisation)
$= 2^2 \times 3^3 \times 5^2$ (Index form)

EXERCISE 2.1

1. Without the calculator, work out the value of the following:

(i) 2^7 (ii) 3^5

(iii) 4^3 (iv) 5^4

(v) 8^3 (vi) 10^5

(vii) $2^3 \times 5^2$ (viii) $3^4 \times 10^3$

2. Express each of the following numbers in index form (exponential form):

(i) $3 \times 3 \times 3 \times 3$

(ii) $7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$

(iii) $a \times a \times a \times a \times a$

(iv) $2.4 \times 2.4 \times 2.4 \times 2.4 \times 2.4 \times 2.4 \times 2.4$

(v) $\dfrac{1}{5 \times 5 \times 5 \times 5 \times 5 \times 5}$

(vi) $\dfrac{1}{11} \times \dfrac{1}{11} \times \dfrac{1}{11} \times \dfrac{1}{11} \times \dfrac{1}{11}$

3. Express each of the following numbers as product of powers of prime numbers:

(i) 720 (ii) 1800

(iii) 5292 (iv) 21600

2.2. LAWS OF INDICES

All numbers written with indices obey the same rule.

Now, we study there basic laws of indices. The word 'indices' is the plural of index.

First Law. If a is a non-zero real number and m, n are positive integers, then

$$a^m \times a^n = a^{m+n}$$

This law also known as the **product law**.

By definition, we have

$$a^m \times a^n = \underbrace{(a \times a \times a \times \ldots \times a)}_{m \text{ factor}} \times \underbrace{(a \times a \times a \times \ldots \times a)}_{n \text{ factor}}$$

$$= a \times a \times a \times \ldots \text{ to } (m + n) \text{ factors}$$

$$= a^{m+n}$$

For example:

(i) $5^4 \times 5^2 = 5^{4+2} = 5^6$

(ii) $\left(\frac{3}{4}\right)^7 \times \left(\frac{3}{4}\right)^5 = \left(\frac{3}{4}\right)^{7+5} = \left(\frac{3}{4}\right)^{12}$

(iii) $x^4 \times x^7 = x^{4+7} = x^{11}$

(iv) $(3x^{-2}y^4)(2x^3y^{-3}) = 6x^{-2+3}y^{4-3} = 6xy$

Second Law. If a is a non-zero real number and m, n are positive integers such that $m > n$, then

$$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$$

This law is also known as the quotient law.

By definition, we have

$$a^m \div a^n = \frac{a^m}{a^n}$$

$$= \frac{a \times a \times a \times \ldots \text{to } m \text{ factors}}{a \times a \times a \times \ldots \text{to } n \text{ factors}}$$

$$= a \times a \times a \times \ldots \text{to} (m - n) \text{ factors} \qquad \text{[Since } n \text{ factors cancel]}$$

$$= a^{m-n}.$$

For example:

(i) $\frac{4^9}{4^6} = 4^{9-6} = 4^3 = 64$

(ii) $\left(\frac{5}{7}\right)^{11} \div \left(\frac{5}{7}\right)^{9} = \left(\frac{5}{7}\right)^{11-9} = \left(\frac{5}{7}\right)^{2}$

(iii) $\frac{x^8}{x^3} = x^8 \div x^3 = x^{8-3} = x^5$

Third Law. If a is a non-zero real number and m, n are positive integers,

then $$(a^m)^n = a^{mn}$$

This law is also known as the **power law.**

By definition, we have

$$(a^m)^n = a^m \times a^m \times a^m \times \ldots \text{to } n \text{ factors}$$
$$= a^{m+m+m+\ldots n \text{ times}}$$
$$= a^{mn}$$

For example:

(i) $(3^4)^5 = 3^{4\times5} = 3^{20}$

(ii) $(y^3)^5 = y^{3\times5} = y^{15}$

EXERCISE 2.2

1. Simplify the following using the law of indices:

(i) $3^4 \times 3^3$ (ii) $2^3 \times 2^5$

(iii) $6^5 \div 6^3$ (iv) $\frac{7^9}{7^6}$

(v) $(4^5)^4$ (vi) $(9^7)^5$

2. Simplify the following:

(i) $5^2 \times 5^3 \times 5^4$ (ii) $(7^2)^3 \div 7^4$

(iii) $(3^{20} \div 3^{16}) \times 3^2$ (iv) $(4^3 \times 4^2)^4$

(v) $(3^2)^5 \times (3^4)^2$ (vi) $((5^2)^3 \times 5^4) \div 5^7$

(vii) $((2^2)^3 \div 2^3) \times (2^3)^4$ (viii) $\left[\left(\frac{2}{3}\right)^7 \times \left(\frac{2}{3}\right)^6\right] \div \left(\left(\frac{2}{3}\right)^3\right)^4$

3. Simplify the following leaving the answer in index form:

(i) $2^3 \times 3^4 \times 2^5 \times 3^2$ (ii) $3^2 \times 5^3 \times 2^3 \times 3^3 \times 7 \times 3^4$

(iii) $2^5 \times 3^4 \times 5^2 \times 2^2 \times 5$ (iv) $2^4 \times 5^2 \times 2^2 \times 5^3 \times 2^3 \times 7$

2.3. PROPERTIES OF INDICES

The following properties of indices in addition to basic three law mentioned earlier are used to simplify expressing involving indices. These are

Property 1. If a is non-zero real number, then $a^0 = 1$

i.e., any non-zero real number raised to the power zero is equal to 1.

Let m be a positive integer, then

$$a^m \div a^m = \frac{a^m}{a^m} \quad \text{(From second law)}$$

$$\Rightarrow \quad a^{m-m} = \frac{a \times a \times a \times \ldots \text{to } m \text{ factors}}{a \times a \times a \times \ldots \text{to } m \text{ factors}}$$

$$\Rightarrow \quad a^0 = 1 \quad \text{(Since all factors cancel)}$$

For example:

$$7^0 = 1, \left(\frac{5}{8}\right)^0 = 1$$

Property 2: (Negative index)

If a is a non-zero real number and m is a positive integer,

then $$a^{-m} = \frac{1}{a^m}$$

Since $$a^0 = 1,$$

we have $$\frac{1}{a^m} = \frac{a^0}{a^m} = a^{0-m} = a^{-m}$$

$$\Rightarrow \quad a^{-m} = \frac{1}{a^m}.$$

Thus, a shift from numerator to denominator or denominator to numerator changes the sign of the exponent.

For example:

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}, \quad 6^{-3} = \frac{1}{6^3} = \frac{1}{216}$$

Note: $a^{-1} = \frac{1}{a}$ *i.e.,*

Any number raised to the power –1 is equal to reciprocal of the number.

For example:

$$2^{-1} = \frac{1}{2} \text{ and } 5^{-1} = \frac{1}{5}$$

Property 3: Power of product of two numbers

If a, b are non-zero real numbers and n is a positive integer,

then $(ab)^n = a^n \times b^n$

By definition, we have

$$(ab)^n = (ab)(ab)(ab)\ldots(ab) \qquad [n \text{ factors}]$$

$$= (a \times a \times a \times \ldots \text{to } n \text{ factors}) \times (b \times b \times b \times \ldots \text{to } n \text{ factors})$$

$$= a^n \times b^n$$

For example:

$$(2 \times 5)^2 = 2^2 \times 5^2 = 4 \times 25 = 100$$

Property 4. Power at Quotients of two numbers

If a, b are non-zero real numbers and n is a positive integer,

then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ provided $b \neq 0$

By definition, we have

$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\ldots\left(\frac{a}{b}\right) \qquad [n \text{ factors}]$$

$$= \frac{a \times a \times a \times \ldots \text{to } n \text{ factors}}{b \times b \times b \times \ldots \text{to } n \text{ factors}}$$

$$= \frac{a^n}{b^n}.$$

For example:

$$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$$

Property 5. Negative power of quotient of two numbers

If a and b are two real numbers and n is postive, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}, \text{ provided } a \neq 0,\ b \neq 0$$

For example:

$$\left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^{3} = \frac{2^3}{3^3} = \frac{8}{27}$$

2.4. RATIONAL POWERS

The index laws (mentioned earlier) can also be applied to rational powers or indices which are written as a fraction.

- $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$

  And $\sqrt{a} \times \sqrt{a} = \sqrt{a \times a} = a$

  $\therefore \quad a^{\frac{1}{2}} = \sqrt{a}$ (By direct comparision)

  i.e., Any number raised to the power $\frac{1}{2}$ is equal to the square root of that number.

For example:

$4^{\frac{1}{2}} = \sqrt{4} = 2, 9^{\frac{1}{2}} = \sqrt{9} = 3, 16^{\frac{1}{2}} = \sqrt{16} = 4, 25^{\frac{1}{2}} = \sqrt{25} = 5$ and $64^{\frac{1}{2}}$ $= \sqrt{64} = 8$.

- $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$

  And $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = \sqrt[3]{a \times a \times a} = a$

  $\therefore \quad a^{\frac{1}{3}} = \sqrt[3]{a}$ (By direct comparison)

  i.e., Any number raised to power $\frac{1}{3}$ is equal to the *cube rule* of that number.

For example:

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2, 27^{\frac{1}{3}} = \sqrt[3]{27} = 3, 64^{\frac{1}{3}} = \sqrt[3]{64} = 4 \text{ and } 125^{\frac{1}{3}} = \sqrt[3]{25} = 5$$

In general, if a is a positive real number, then

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad \text{(where } n > 0\text{) ...(1)}$$

i.e., Any number raised to the power $\frac{1}{n}$ is equal tot the nth root of that number.

Also, notice that $a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{12}{4}} = a^3$

$$\left(a^{\frac{3}{4}}\right)^4 = a^3$$

$$a^{\frac{3}{4}} = \sqrt[4]{a^3}$$

In general, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ or $\sqrt[n]{a^m}$ ...(2)

i.e., if the power if a number a > 0 is fraction of the form $\frac{m}{n}$, then it is called rational power of a number.

For example:

$$64^{\frac{2}{3}} = (\sqrt[3]{4})^2 = 4^2 = 16$$

Example 3: *Simplify the following:*

(i) $(16)^{\frac{3}{4}}$ (ii) $(25)^{\frac{-3}{2}}$

(iii) $\left(\frac{100}{9}\right)^{\frac{-1}{2}}$ (iv) $\left(\frac{125}{64}\right)^{\frac{-2}{3}}$

Solution:

(i) $(16)^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^{4\times\frac{3}{4}} = 2^3 = 8$

(ii) $(25)^{\frac{-3}{2}} = (5^2)^{\frac{-3}{2}} = 5^{2\times\frac{-3}{5}} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

(iii) $\left(\frac{100}{9}\right)^{\frac{-1}{2}} = \left(\frac{9}{100}\right)^{\frac{1}{2}} = \left(\frac{3^2}{10^2}\right)^{\frac{1}{2}} = \frac{(3^2)^{\frac{1}{2}}}{(10^2)^{\frac{1}{2}}} = \frac{3^{2\times\frac{1}{2}}}{10^{2\times\frac{1}{2}}} = \frac{3}{10}$

(iv) $\left(\frac{125}{64}\right)^{\frac{-2}{3}} = \left(\frac{64}{125}\right)^{\frac{2}{3}} = \left(\frac{4^3}{5^3}\right)^{\frac{2}{3}} = \frac{(4^3)^{\frac{2}{3}}}{(5^3)^{\frac{2}{3}}} = \frac{4^{3\times\frac{2}{3}}}{5^{3\times\frac{2}{3}}} = \frac{4^2}{5^2} = \frac{16}{25}$

Example 4: *Simplify the following.*

(i) $64^{\frac{-1}{3}}\left(64^{\frac{1}{3}} - 64^{\frac{2}{3}}\right)$ (ii) $\left(81^{\frac{1}{4}} + 256^{\frac{1}{4}}\right) \div \left(32^{\frac{1}{5}} + 125^{\frac{1}{3}}\right)$

(iii) $\dfrac{(49)^{\frac{2}{3}} \div (49)^{\frac{1}{6}}}{\left(\frac{1}{7}\right)^{\frac{-7}{6}} \times \left(\frac{1}{7}\right)^{\frac{1}{6}}}$

Solution:

(i) $64^{\frac{-1}{3}}\left(64^{\frac{1}{3}} - 64^{\frac{2}{3}}\right)$

$$= (4^3)^{\frac{-1}{3}}\left[(4^3)^{\frac{1}{3}} - (4^3)^{\frac{2}{3}}\right]$$

$$= 4^{3\times\left(\frac{1}{-3}\right)}\left[4^{3\times\frac{1}{3}} - 4^{3\times\frac{2}{3}}\right]$$

$$= 4^{-1}\left(4 - 4^2\right) = \frac{1}{4}(4 - 16) = \frac{1}{4}(-12) = -3$$

(ii) $\left(81^{\frac{1}{4}} + 256^{\frac{1}{4}}\right) \div \left(32^{\frac{1}{5}} + 125^{\frac{1}{3}}\right)$

$$= \left[(3^4)^{\frac{1}{4}} + (4^4)^{\frac{1}{4}}\right] \div \left[(2^5)^{\frac{1}{5}} + (5^3)^{\frac{1}{3}}\right]$$

$$= (3 + 4) \div (2 + 5) = 7 \div 7 = \frac{7}{7} = 1$$

(iii) $$\frac{49^{\frac{2}{3}} \div 49^{\frac{1}{6}}}{\left(\frac{1}{7}\right)^{-\frac{7}{6}} \times \left(\frac{1}{7}\right)^{\frac{1}{6}}} = \frac{\left(7^2\right)^{\frac{2}{3}} \div \left(7^2\right)^{\frac{1}{6}}}{\left(\frac{1}{7}\right)^{\frac{-7}{6}+\frac{1}{6}}} = \frac{7^{\frac{4}{3}} \div 7^{\frac{1}{3}}}{\left(\frac{1}{7}\right)^{-1}} = \frac{7^{\frac{4}{3}-\frac{1}{3}}}{(7)^1} = \frac{7^1}{7^1} = 7\ 1$$

Example 5: *Simplify the following:*

$$\left[\left\{\left(-\frac{1}{2}\right)^{-2}\right\}^{2}\right]^{-2}$$

Solution: We have

$$\left[\left\{\left(-\frac{1}{2}\right)^{-2}\right\}^{2}\right]^{-2} = \left\{\left(-\frac{1}{2}\right)^{-2}\right\}^{2\times(-2)}$$

$$= \left\{\left(-\frac{1}{2}\right)^{-2}\right\}^{-4}$$

$$= \left(-\frac{1}{2}\right)^{(-2)\times(-4)}$$

$$= \left(\frac{-1}{2}\right)^{8} = \frac{(-1)^8}{2^8}$$

$$= \frac{1}{256}$$

Example 6: *Simplify the following, Assuming that x is a positive real number and a, b, c are rational numbers,*

$$\left(\frac{x^b}{x^c}\right)^{a}\left(\frac{x^c}{x^a}\right)^{b}\left(\frac{x^a}{x^b}\right)^{c}$$

Solution: We have,

$$\left(\frac{x^b}{x^c}\right)^a\left(\frac{x^c}{x^a}\right)^b\left(\frac{x^a}{x^b}\right)^c = (x^{b-c})^a(c^{c-a})^b(x^{a-b})^c$$

$$= x^{ab-ac} \times x^{bc-ab} \times x^{ac-bc}$$

$$= x^{ab-ac+bc-ab+ac-bc} = x^0 = 1$$

EXERCISE 2.3

1. Simplify the following:

(i) $5^2 \times 5^3 \times 5^{-4}$

(ii) $(7^2)^{-3} \div 7^{-4}$

(iii) $(3^{20} \div 3^{14}) \times 3^{-3}$

(iv) $\left(4^{\frac{1}{3}} \times 8^{\frac{-1}{3}}\right)^{-3}$

(v) $(3^{-2})^5 \times (3^4)^2$

(vi) $\left[(5^{-2})^3 \times 5^4\right] \div (5)^{-6}$

(vii) $\left(\frac{2}{11}\right)^{-3} \times \left(\frac{11}{3}\right)^{\frac{-1}{3}} \times \left(\frac{3}{2}\right)^{-3}$

2. Simplify the following:

(i) $32^{\frac{-2}{6}}$

(ii) $16^{\frac{-3}{2}}$

(iii) $100^{1\frac{1}{2}}$

(iv) $\left(\frac{81}{16}\right)^{\frac{-1}{2}}$

(v) $\left(\frac{64}{24}\right)^{\frac{-1}{2}}$

(vi) $\left(\frac{8}{27}\right)^{\frac{-2}{3}}$

3. Simplify the following:

(i) $(81 \times 3^{-3})(16 \times 2^{-4})$

(ii) $(225^{\frac{1}{2}} + 70) \times 256^{-\frac{1}{4}}$

(iii) $\frac{7^6 \times 7^{-2}}{7^5}$

4. Simplify:

(i) $\dfrac{2^4 \times 3^5 \times 8}{32 \times 27}$ (ii) $\dfrac{(49)^2 \times (25)^3 \times (12)^4}{7^3 \times (30)^6}$

(iii) $\left(\dfrac{4}{9}\right)^{\frac{1}{2}}\left(\dfrac{16}{81}\right)^{-\frac{1}{4}}\left(\dfrac{32}{243}\right)^{\frac{1}{5}}$ (iv) $\dfrac{5^{n+2} - 5^{n+1}}{5^{n+3}}$

5. Simplify:

(i) $\left(\dfrac{x^a}{y^b}\right)^{a+b} \times \left(\dfrac{x^b}{x^c}\right)^{b+c} \times \left(\dfrac{x^c}{x^a}\right)^{c+a}$

(ii) $\left(\dfrac{x^a}{a^b}\right)^{\frac{1}{ab}} \times \left(\dfrac{x_b}{x^c}\right)^{\frac{1}{bc}} \times \left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}}$

(iii) $\sqrt[p]{\dfrac{x^a}{x^b}} \times \sqrt[p]{\dfrac{x^b}{x^c}} \times \sqrt[p]{\dfrac{x^c}{x^a}}$

6. If $4^{-n} = x$ find 2^{2n}

7. If $3^n = k$ find 3^{n+2}

2.5. EXPONENTIAL GROWTH

Exponential growth is a process that increases quantity with passage of time, creating the curve of an exponential function. If a quantity is increasing exponentially, then the quantity is said to have **exponential growth**.

In this section, we will study situations where the quantities are increasing exponentially. These situations are known as **growth** and occur frequently in the real world around us. For example, populations of bacteria, animals, and people usually grow exponentially.

The function $y = a^x$ is an exponential function. In general, the exponential growth of a quantity P is represented by the following equation as:

$$P_t = P_0 \times a^{kt},$$

where,

P_t = Value of Growth of P after t time

P_0 = Initial value of P at time $t = 0$

a = Constant, called positive growth factor such that $a > 1$

k = Constant and $k > 0$ and

t = Time

Consider a population of bacteria in a culture is 5000, which is increasing by 10% each day under favourable conditions. Express this situation mathematically. To increase a quantity by 5%, we multiply it by 110% or 1.1.

If P_t is the population of bacteria after t days, then

$P_0 = 5000$ (Initial or original population of bacteria)

$P_1 = P_0 \times 1.1 = 5000 \times 1.1$ (Population of bacteria after 1 day)

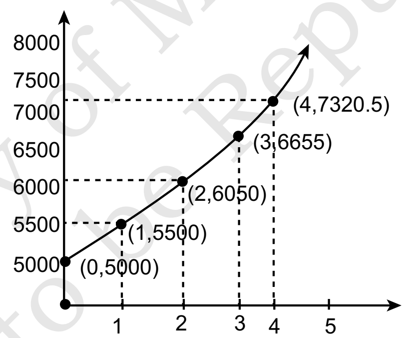
$P_2 = P_1 \times 1.1 = P_0 \times (1.1)^2 = 5000 \times (1.1)^2$ (Population of bacteria after 2 days)

$P_3 = P_2 \times 1.1 = P_0 \times (1.1)^3 = 5000 \times (1.1)^3$ (Population of bacteria after 3 days), and so on.

From the pattern,

$P_t = P_0 \times (1.1)^t = 5000 \times (1.1)^t$ (Population of bacteria after t days)

This is the mathematical equation for the given situation.



Example 7: *The population of bacteria in a culture is given by:* $P_t = 2000 \times 2^{0.1t}$*, where t is the time in hours. Find*

(i) Initial population

(ii) Population after 5 hours

(iii) Population after 10 hours

(iv) Population after 14 hours

(v) Draw the graph of P_t against t.

Solution:

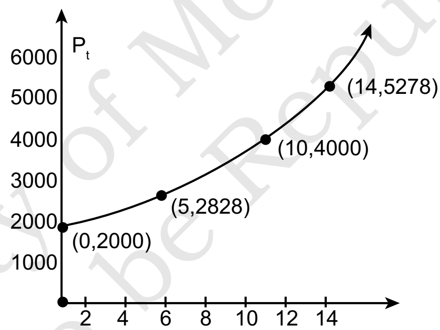
(i) Initial population, when $t = 0$:
$P_0 = 2000 \times 2^{0.1 \times 0} = 2000 \times 2^0 = 2000$
The initial population of bacteria was 2000.

(ii) Population after 5 hours i.e., when $t = 5$:
$P_5 = 2000 \times 2^{0.1 \times 5} = 2000 \times 2^{0.5} = 2828$ (approx)
The population of bacteria after 5 hours is 2828.

(iii) Population after 10 hours i.e., when $t = 10$:
$P_{10} = 2000 \times 2^{0.1 \times 10} = 2000 \times 2^1 = 4000$
The population of bacteria after 10 hours is 4000.

(iv) Population after 14 hours i.e., when $t = 14$:
$P_{14} = 2000 \times 2^{0.1 \times 14} = 4000 \times 2^1 = 5278$ (approx)
The population of bacteria after 14 hours is 5278.

(v) Graph of function is shown below:



EXERCISE 2.4

1. *A population of mice which is under favourable conditions is increasing as: $P_t = 100 \times (1.2)^t$, where t is the time in weeks. Find*

(i) Initial population
(ii) Population after 1 week
(iii) Population after 3 weeks
(iv) Population after 6 weeks
(v) Draw the graph of P_t against t.

2. *In a region, an entomologist records the affected area by the grasshoppers. The area affected by the grasshoppers is given by: $A_n = 1500 \times 2^{0.2n}$ hectares, where n is the number of weeks after the initial observation. Find*

(i) Initial affected area

(ii) Affected area after 5 weeks

(iii) Affected area after 10 weeks

(iv) Population after 15 weeks

(v) Draw the graph of A_n against n.

2.6. LOGARITHMS

When we given the base 2 and exponent 3, we can calculate $2^3 = 8$.

Inversely, if we are given base 2 and its power 8, and we need to find the power or index or exponent, $2^? = 8$. This exponent is called the **logarithm**. In this case, we call the power 3 as the 'logarithm of 8 with base 2'. We write it as: $3 = \log_2 8$

3 is the index or power to which 2 must be raised in order to obtain 8.

Instead of writing $2^3 = 8$, we can also say that '3 is the logarithm of 8 to base 2'. Simply, we write logarithm as **log**. Therefore,

$$2^3 = 8 \Rightarrow \log_2 8 = 3$$

Similarly, $$5^4 = 625 \Rightarrow \log_5 625 = 4$$

And $$10^2 = 100 \Rightarrow \log_{10} 100 = 2$$

Definition: If $a^x = b$, then the logarithm of a number b to the base a (i.e., $\log_a b$) is x i.e., the index or the power to which the base a must be raised in order to obtain the given number b. From the definition,

$$a^x = b \Rightarrow \log_a b = x$$

provided $a > 0$ (i.e, a is positive), $a \neq 1$ and $b > 0$.

Here, $a^x = b$ is the *exponential form* and its equivalent *logarithmic form* is $\log_a b = x$ and vice versa

Note:

1. The logarithm of a negative number does not exist.
2. The base of logarithm can be written in any base except 1.
3. When base is 10, then the logarithm is called **common logarithm** and when the base is 'e', then it is called **natural logarithm**.
4. If the base of a logarithm is not written, it means it is in the base 10.

Properties of Logarithms

Property 1. $\log_a 1 = 0$ i.e., the logarithm of 1 to any base is always zero..

Since $a^0 = 1$, therefore by definition of log, $\log_a 1 = 0$

For example:

$\log_2 1 = 0$, $\log_3 1 = 0$, $\log_4 1 = 0$, etc.

Property 2. $\log_a a = 1$ i.e., the logarithm of any positive number to the same base is always one.

Since $a^1 = a$, therefore by definition of log, $\log_a a = 1$

For example:

$\log_2 2 = 1$, $\log_3 3 = 1$, $\log_4 4 = 0$, $\log_{10} 10 = 0$ etc.

Property 3. If $y = a^x$, then $x = \log_a y$ and so $x = \log_a a^x$

For example: $3 = \log_2 2^3$

Property 4. If $x = a^y$, then $y = \log_a x$

and so

$x = a^{\log_a x}, x > 0$

For example: $5 = 3^{\log_3 5}$

Example 8: *Write an equivalent logarithmic form for each the following:*

(i) $2^5 = 32$ (ii) $3^4 = 81$ (iii) $1000 = 10^3$

(iv) $7^0 = 1$ (v) $81^{\frac{1}{4}} = 3$ (vi) $10^{-2} = 0.01$

(vii) $25 = x^3$ (viii) $128^{\frac{1}{7}} = 2$

Solution:

If follow from the definition of logarithm that:

(i) $2^5 = 32 \Rightarrow \log_2 32 = 5$

(ii) $3^4 = 81 \Rightarrow \log_3 81 = 4$

(iii) $1000 = 10^3 \Rightarrow \log_{10} 1000 = 3$

(iv) $7^0 = 1 \Rightarrow \log_7 1 = 0$

(v) $81^{\frac{1}{4}} = 3 \Rightarrow \log_{81} 3 = \frac{1}{4}$

(vi) $10^{-2} = 0.01 \Rightarrow \log_{10} 0.01 = -2$

(vii) $25 = x^3 \Rightarrow \log_x 25 = 3$

(viii) $128^{\frac{1}{7}} = 2 \Rightarrow \log_{128} 2 = \frac{1}{7}$

Example 9: *Convert the following into exponential form:*

(i) $\log_5 125 = 3$ (ii) $\log_3 243 = 5$

(iii) $\log_{10} 0.001 = -3$ (iv) $\log_{25} 25 = 1$

(v) $\log_{64} 4 = \frac{1}{3}$ (vi) $\log_5 \frac{1}{5} = -1$

Solution: It also follows from the definition at logarithm that:

(i) $\log_5 125 = 3 \Rightarrow 5^3 = 125$

(ii) $\log_3 243 = 5 \Rightarrow 3^5 = 243$

(iii) $\log_{10} 0.001 = -3 \Rightarrow 10^{-3} = 0.001$

(iv) $\log_{25} 25 = 1 \Rightarrow 25^1 = 25$

(v) $\log_{64} 4 = \frac{1}{3} \Rightarrow 64^{\frac{1}{3}} = 4$

(vi) $\log_5 \frac{1}{5} = -1 \Rightarrow 5^{-1} = \frac{1}{5}$

Example 10: *Find the values of each of the following:*

(i) $\log_2 64$ (ii) $\log_5 625$

(iii) $\log_{16} 2$ (iv) $\log_{10} 0.01$

Solution:

(i) Let $\log_2 64 = x$. Then

$\log_2 64 = x \Rightarrow 2^x = 64 \Rightarrow 2^x = 2^6 \Rightarrow x = 6$

$\therefore \log_2 64 = 6$

(ii) Let $\log_5 625 = x$. Then

$\log_5 625 = x \Rightarrow 5^x = 625 \Rightarrow 5^x = 5^4 \Rightarrow x = 4$

$\therefore \log_5 625 = 4$

(iii) Let $\log_{16} 2 = x$. Then

$\log_{16} 2 = x \Rightarrow 16^x = 2 \Rightarrow (2^4)^x = 2 \Rightarrow 2^{4x} = 2^1$

$\Rightarrow 4x = 1 \Rightarrow x = \dfrac{1}{4}$

$\therefore \log_{16} 2 = \dfrac{1}{4}$

(iv) Let $\log_{10} 0.01 = x$. Then

$\log_{10} 0.01\ x \Rightarrow 10^x = 0.01 \Rightarrow 10^x = \dfrac{1}{100} \Rightarrow 10^x = \dfrac{1}{10^2}$

$\Rightarrow 10^x = 10^{-2} \Rightarrow x = -2$

$\therefore \log_{10} 0.01 = -2$

Example 11: *If $\log_2 y = x$, find the value of 8^x in terms of y.*

Solution: We have, $\log_2 y = x$

$\Rightarrow 2^x = y$

$\therefore 8^x = (2^3)^x = 2^{3x} = (2x)^3 = y^3$

EXERCISE 2.5

1. Express each of the following in logarithmic form:

(i) $3^5 = 243$ (ii) $10^4 = 10000$

(iii) $125^{-\frac{1}{3}} = 5$ (iv) $10^{-3} = 0.001$

(v) $(1.2)^2 = 1.44$ (vi) $64^{\frac{2}{3}} = 16$

2. Express each of the following in exponential form:

(i) $\log_4 1024 = 5$ (ii) $\log_2 512 = 9$

(iii) $\log_{81} 3 = \dfrac{1}{4}$ (iv) $\log_{10} 0.0001 = -4$

(v) $\log_{343} 49 = \dfrac{2}{3}$ (vi) $\log_{\sqrt{2}} 4 = 4$

3. Find the values of each of the following

(i) $\log_9 81$ (ii) $\log_{16} 64$

(iii) $\log_2\left(\dfrac{1}{32}\right)$ (iv) $\log_5\left(\dfrac{1}{625}\right)$

(v) $\log_2 0.125$ (vi) $\log_8 512$

4. If $\log_{10} y = x$, find the value of 10^{2x} in terms of y.

5. If $\log_{10} 4 = a$, find the value of 10^{2a-1} in terms of x.

6. If $\log_3 x = a$, find the value of 81^{a-1} in terms of x.

7. Find the values of x in each of the following:

(i) $\log_2 x = 3$ (ii) $\log_9 x = 2.5$

(iii) $\log_{81} x = \frac{3}{2}$ (iv) $\log_{\sqrt{2}} x = 4$

2.7. LOGARITHM FUNCTIONS

If the inverse of the exponential function ($y = a^x$) exists, then we can represent the logarithmic function as given below.

The logarithmic function is given by:

$$y = f(x) = \log_a x \text{ if } a^y = x$$

where x is variable and a is constant which is called the base of function such that $a > 1$.

If the base $a = 10$, then it is called a common logarithm and if $a = e$ (value of e is equal to 2.71828), then it is called the natural logarithm. Generally, the natural logarithm is denoted by *ln*.

2.8. LOGARITHMS IN BASE 10

We know that $10^0 = 1$

$10^1 = 10$

$10^2 = 100$

$10^3 = 1000$

....

LHS is the exponential form of the number on RHS. 10^3 is the exponential form of 1000. Here. 10 is the base and 3 is the exponent (or index or power). The third power of 10 is 1000.

Now consider the statements.

Which power of 10 is 1000?

Or what is the exponent to which the base 10 must be raised to get 1000?

In mathematical symbols, these equivalent statements are written as $\log_{10} 1000$. Here, log is an abbreviation for logarithm which means a rule of arithmetic. It is read as 'logarithm of 1000 in the base 10.'

Thus, $\log_{10} 1000$ = the exponent of base 10 which gives 1000 = 3

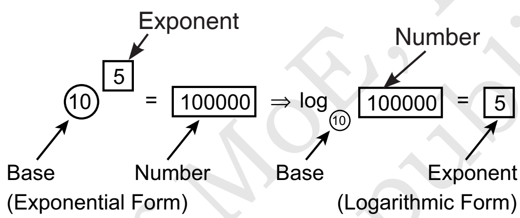
Logarithms are exponents. When the base is 10, the logarithms are called common logarithms. Since we have to study only common logarithms, the base 10 may be dropped. Thus, $\log_{10} x$ may be written as $\log x$, where

$\log x$ = the exponent of 10 which gives x.

log100 = the exponent of 10 which gives 100 = 2, since $10^2 = 100$

Now, for all real values of n, positive, negative or zero, $10^n = x$ is always positive. Therefore, $\log x$ is meaningful only when $x > 0$. We cannot find logarithms of negative numbers.

Every logarithmic form can be translated into an exponential form and vice versa. Thus,



and $\log_{10} 100000 = 5 \Rightarrow 10^5 = 100000$

(Logarithmic Form) (Exponential Form)

In general, $10^y = x \Rightarrow \log_{10} x = y$ and $\log_{10} x = y \Rightarrow 10^y = x$

Example 12: *Without using a caculator, find*

(i) $\log 10000$ (ii) $\log \sqrt[4]{1000}$

(iii) $\log\left(\dfrac{100}{\sqrt{10}}\right)$ (iv) $\log(10^a \times 10^b)$

Solution:

(i) $\log 10000 = \log 10^4 = 4 \log 10 = 4$ ($\because \log 10 = 1$)

(ii) $\log \sqrt[4]{1000} = \log(1000)^{\frac{1}{4}} = \log\left(10^3\right)^{\frac{1}{4}} = \log 10^{\frac{3}{4}}$

$$= \frac{3}{4}\log 10 = \frac{3}{4} (\because \log 10 = 1)$$

(iii) $\log\left(\frac{100}{\sqrt{10}}\right) = \log\left(\frac{10^2}{10^{\frac{1}{2}}}\right) = \log\left(10^2 \times 10^{\frac{-1}{2}}\right)$

$= \log\left(10^{2-\frac{1}{2}}\right) = \log 10^{\frac{3}{2}}$

$= \frac{3}{2}\log 10 = \frac{3}{2}$

(iv) $\log(10^a + 10^b) = \log(10^{a+b}) = (a+b)\log 10 = a + b$

Representation of positive number in the form 10x using logarithms in base 10

In fact, all positive numbers can be written in the form 10^x by using logarithms in base 10.

Many positive numbers can be easily written in the form 10^x

Base 10 logarithm **(logarithm in base 10)** of a positive number is the power to which 10 is raised to give that number. (By definition of logarithm)

For example:

- Since $1000 = 10^3$, we write $\log_{10} 1000 = 3$ or $\log 1000 = 3$
- Since $0.01 = 10^{-2}$, we write $\log_{10}(0.01) = -2$, or $\log(0.01) = -2$.

$\therefore$ $a = 10^{\log a}$ for any $a > 0$.

It is the representation of a positive number in the form 10^x by using logarithms in base 10.

Example 13: *Using calculator, write the following numbers in the form 10^x, where x is correct to 4 decimal places:*

(i) 6 (ii) 15

(iii) 0.04 (iv) 400

Solution:

Using the formula, $a = 10^{\log a}$, for any $a > 0$.

(i) $6 = 10^{\log 6} \approx 10^{0.7782}$ (By Calculator)

(ii) $15 = 10^{\log 15} \approx 10^{1.1761}$ (By Calculator)

(iii) $0.04 = 10^{\log 0.04} \approx 10^{-1.3979}$ (By Calculator)

(iv) $400 = 10^{\log 400} \approx 2.6021$ (By Calculator)

Example 14: Find x if:

(i) $\log x = 4$

(ii) $\log x = -\frac{1}{3}$

(iii) $\log x \approx -1.378$

Solution:

(i) $x = 10^{\log x}$

$\Rightarrow x = 10^4$ $(\because \log x = 4)$

$= 10000$

(ii) $x = 10^{\log x}$

$\Rightarrow x = 10^{-\frac{1}{3}}$ $\left(\because \log x = -\frac{1}{3}\right)$

$= \frac{1}{1000} = 0.001$

(iii) $x = 10^{\log x}$

$\Rightarrow x \approx 10^{-1.378}$

$\Rightarrow x \approx 0.0419$

EXERCISE 2.6

1. Write an equivalent logarithmic form in base 10 for each of the following:

(i) $10^4 = 10000$

(ii) $10^{-2} = 0.01$

(iii) $(100)^{2.5} = 100000$

(iv) $10^{0\cdot5} = 3.1623$

(v) $(1000)^{-3} = 0.1$

(vi) $(\sqrt{10})^6 = 1000$

2. Write an equivalent exponential form for each of the following:

(i) $\log_{10} 100 = 2$

(ii) $\log_{10} 0.1 = -1$

(iii) $\log 0.001 = -3$

(iv) $\log 1000 = 3.$

3. Without using a calculator, simplify the following:

(i) $\log_{10} 100000$ (ii) $\log_{10} 0.0001$

(iii) $\log_{10} \sqrt[5]{100}$ (iv) $\log_{10} \left(100\sqrt[3]{10}\right)$

(v) $\log_{10} \left(\dfrac{10^{a-\frac{1}{3}}}{\sqrt[3]{100}}\right)$ (vi) $\log_{10} \left(\dfrac{100^a}{100^b}\right)$

4. Using the calculator, write to the following numbers in the form 10^x, where x is correct to 4 decimal places:

(i) 8 (ii) 60

(iii) 800 (iv) 1.5

(v) 0.015 (vi) 0.005

5. Find the values of x in each of the following:

(i) $\log_{10} x = 5$ (ii) $\log_{10} x = -4$

(iii) $\log_{10} x = \dfrac{1}{2}$ (iv) $\log_{10} x = -\dfrac{1}{3}$

(v) $\log_{10} x \approx 1.0792$ (vi) $\log_{10} x \approx -3.1997$

2.9. LOGARITHMIC NUMBERS GREATER THAN 10

So far, we have evaluated some logarithms by inspection, using the definition that logarithms are exponents. But that does not help in general. We cannot find $\log_{10} 2$ or $\log_{10} 7$ by inspection. There are two methods for finding the logarithms of numbers.

Method I. Using a calculator

A calculator with a LOG key can be used to evaluate common logarithms.

For example:

Logarithm	Calculator Keystrokes			Display
log17	LOG	17	ENTER	1.2304
log29	LOG	29	ENTER	1.4624
log15.79	LOG	15.79	ENTER	1.1984
log43.86	LOG	43.86	ENTER	1.6421

Method II. Using table of common logarithms

We know that every positive real number x can be expressed in the standard form a × 10^n where 1 C $a < 10$ and n is an integer,

$$
\begin{aligned}
\text{i.e.,} \quad x &= a \times 10^n \\
\Rightarrow \quad \log x &= \log (a \times 10^n) \\
&= \log a + \log 10^n \quad \text{(By product rule)} \\
&= \log a + n \log 10 \quad \text{(By power rule)} \\
&= \log a + n \times 1 \\
&= n + \log a \qquad \ldots(1) \\
\text{Now,} \quad \log 1 &= 0, \log 10 = 1 \\
\text{and} \quad 1 &\leq a < 10 \\
\Rightarrow \quad \log 1 &\leq \log a < \log 10 \\
\text{or} \quad 0 &\leq \log a < 1 \qquad \ldots(2)
\end{aligned}
$$

From (1) and (2), we conclude that the **logarithm of a positive real number x consists of the sum of two (parts) numbers**, (i) **an integer part n and decimal part long a, which lies between 0 and 1**. The integer decimal part is called the characteristic and the decimal part, the number between 0 and 1 is called the **mantissa**.

The mantissa in never negative and always less than one.

Thus, logarithm of x (> 10) is given by:

∴ log x = **Characteristic** + **Mantissa**

↑ Positive integral part ↑ Decimal part

For the numbers greater than 10,

(i) the characteristic is positive and greater than 1

(ii) the mantissa is always positive

(iii) the value of logarithm is always positive

The **characteristic of number is the index or exponent of 10 in the standard form of that number**.

For example:

(i) $3862 = 3.862 \times 10^3$

⇒ Characteristic of log 3862 is 3.

(ii) $75.48 = 7.548 \times 10^1$

⇒ Characteristic of log 75.48 is 1.

In general, the characteristic of logarithm of a number greater than one is one less than the number of digits i.e., the number of digits to the left of decimal point.

The mantissa is read from the four figure table of common logarithms as follows:

(i) Omit the decimal point from the given number. We consider first form digits from the left most side of the number. If the number is of one digit only, add 0 to its right and make it a two-digit number. (Mantissa of log 5 is same as mantissa of log 50).

(ii) In the table of logarithms, run your eye down the extreme left-hand column till you arrive at the first two digits of the number from the left. The mantissa of the logarithm of two-digit number is the entry at the intersection of row headed by the two-digit number and the column headed by 0, prefixed with the decimal point.

For example:

Mantissa of log 37 is 5682 pre-fixed with the decimal point, i.e., 0.5682.

	0	…
⋮	↓	
37	5682	…

(iii) If the number is of three digits, run your eye down the extreme left-hand column till you arrive at the first two digits of the number from the left. The mantissa of the logarithm of three-digit number is the entry at the intersection of row headed by the first two digits of the number and the column headed by the third digit of the number, pre-fixed with the decimal point.

For example:

Mantissa of log 374 is 5729 pre-fixed with the decimal point, i.e., 0.5729

	0	1	2	3	4	...
...					↓	
37					5729	...

(iv) If the number is of four digits, trace the fourth digit in the proportional parts table at the extreme right of the common logarithm table. Run your eye vertically down the digit and horizontally along the row in which the number in step (iii) lies. Find the number situated at the intersection of the column and the row. Add this number to the number in step (iii). Prefix a decimal point to this number.

For example:

Mantissa of log 3745 is 5729 + 6 = 5735 prefixed with the decimal point, i.e., 0.5735

	0	1	2	3	4	...	1 2 3 4 5 ...
...					↓		↓
37					5729 →	+ →	6

Example 15: *Write the characteristics of the logarithms of each of the following numbers:*

(i) 20 (ii) 459.72 (iii) 2314.2

Solution:

(i) The given number in standard form is within as:

$20 = 2 \times 10^1$

$\Rightarrow$ Characteristic of log 20 is 1 i.e., index of 10 in standard form

(ii) The given number in standard form is written as:

$459.72 = 4.5972 \times 10^2$

$\Rightarrow$ Characteristic of log 459.72 is 2.

(iii) The given number in standard from is written as:

$2314.2 = 2.3142 \times 10^3$

$\Rightarrow$ Characteristic of log 2314.2 is 3.

Example 16: *Use logarithm tables to find the logarithm of each of the following numbers. Use calculator to verify your result.*

(i) 25.795 (ii) 100.76

Solution:

(i) We have, $25.795 = 2.5795 \times 10^1$

Characteristic of the logarithm is 1.

To find mantissa of log 25.795, we take the first four digits (after omitting the decimal point). The number formed by first four digits is 2579. Now, we look in the row standing with 25. In this row, look at the number in the column headed by 7. The number is 4099. Now, more to the column of mean difference on the right and look under the column headed by 9 in the row corresponding to 25. We find the number 15. Add this number to 4099. We get the number 4114. Thus, the required mantissa is 4114. Therefore log 25.795 = 1 + 0.4114 = 1.4114

Note: To avoid the process of finding mantissa, we can directly write logarithm of given number as:

1og 25.795 = Characteristic + Mantissa 1 + 0.4099 (+ 15 added at the end)

= 1 + 0.4114

= 1.4114

	0	1	2		7	...	1 2 3 ... 9
...					↓		↓
25					4099	→	→ 15

(ii) We have, $100.76 = 1.00 \times 10^2$

Characteristics of log 100.76 is 2.

To find mantissa of log 100.76, we take first four significant digits i.e., the number 1007. Now take first two digits i.e., 10 and look in the row starting with 10. In this row, look at the number in the column headed 0. The number is 0000. Now, more the column of mean differences on the right and look under the column headed by 7 in the row corresponding to 10. we find the number 29. Add this number to 0000 to get the required mantissa. Thus, required mantissa is 0029.

Note: Write mantissa always in four figure.

	0		1 2 3 … 7 …
10 →	0.000	⟶	↓ 29

$\therefore$ log 100.76 = 2 + 0.0029 = 2.0029

We can also write as:

log 100.76 = Characteristic + Mantissa = 2 + 0.0000 (+ 29 added at the end)

= 2.0.0029

= 2.0029

Check the result by calculator:

log 100.76 = 2.0033

2.10. LOGARITHMIC OF NUMBERS BETWEEN 0 AND 1

There are two methods for finding the logarithms of numbers between 0 and 1.

Method I. Using a Calculator

A calculator with a LOG key can be used to evaluate common logarithms.

For example:

Logarithm	Calculator Keystrokes	Display
log 0.0124	LOG 0.0124 ENTER	–1.9066
log 0.7523	LOG 0.7523 ENTER	–0.1236
log 0.0024	LOG 0.0024 ENTER	–2.6198
log 0.00036	LOG 0.00036 ENTER	–3.4437

Method II. Using table of common logarithms

In similar way as we have done in earlier section, we express the given number x, which is between 0 and 1, in the standard form $a \times 10^{-n}$ where $1 \le a < 10$ and n is a positive integer,

i.e., $x = a \times 10^{-n}$

$\Rightarrow$ $\log x = \log (a \times 10^{-n})$

$= \log a + \log 10^{-n}$ (By product rule)

$= \log a - n \log 10$ (By power rule)

$= \log a - n \times 1$

$= -n + \log a$ ...(1)

Now, $\log 1 = 0, \log 10 = 1$

and $1 \le a < 10$

$\Rightarrow$ $\log 1 \le \log a < \log 10$

or $0 \le \log a < 1$ ...(2)

From (1) and (2), we conclude that the logarithm of a number x consists of the sum of two parts integer part $-n$ and decimal part log a, which lies between 0 and 1. The integer part is called the **characteristic** and the decimal part the number between 0 and 1 is called the **mantissa**.

The characteristic of a number less than 1 is the exponent of 10 in the stnadard form of the number.

For example:

(i) $0.003862 = 3.862 \times 10^{-3}$

$\Rightarrow$ Characteristic of log 0.003862 is –3.

(ii) $0.7548 = 7.548 \times 10^{-1}$

$\Rightarrow$ Characteristic of log 0.7548 is –1.

In general, the characteristic of logarithm of a number less than 1 is negative of one less the number of zeros between the decimal point and the first significant (non-zero) digit.

i.e., negative of the number of decimal places the first significant occurs.

The mantissa of a number less than 1 is read from the four figure table of common logarithms as follows:

(i) Omit the decimal point from the given number. We consider first four significant digits from the left most side of the given number. If the number has one digit to the right of the decimal point, add '0' to its right and make it a two-digit number. For example, to find mantissa of 0.3, we consider the number 50. Also, if the number has for or more consecutive zeros to the right of the decimal point, then its mantissa is calculated with the help of the number formed by digits beginning with the non-zero digit. For example, to find mantissa of 0.000023058, we consider the number 2305.

(ii) In the table of logarithms, run your eye down the extreme left-hand column till you arrive at the first two digits of the number from the left. The mantissa of the logarithm of a number having two digit right of the decimal point is the entry at the intersection of row headed by the two-digit number and the column headed by 0, prefixed with the decimal point.

For example:

Mantissa of log 0.37 is 5682 pre-fixed with the decimal point, i.e., 0.5682.

Note: Mantissa of log 0.37, log 0.037 and log 0.0037 is same.

	0	...
⋮	↓	
37	5682	...

(iii) If the number has three digits right of the decimal point, run your eye down the extreme left-hand column till you arrive at the first two digits of the number from the left. The mantissa of the logarithm of this number is the entry at the intersection of row headed by the first two digits and the column headed by the third digit, pre-fixed with the decimal point.

Thus, logarithm of x (<1) is given by:

$\log x = -n + \log a$

= **Characteristic** + **Mantissa**

↓ ↓

Negative Integral part   Decimal part

For example:

Mantissa of log 0.374 is 5729 pre-fixed with the decimal point, i.e., 0.5729.

Note: Mantissa of log 0.374, log 0.0374 and log 0.000374 is same.

	0	1	2	3	4	...
...					↓	
37					5729	...

(iv) If the number has four digits right of the decimal point, trace the fourth digit in the proportional parts table at the extreme right of the common logarithm table, Run your eye vertically down the digit and horizontally along the row in which the number in step (iii) lies. Find the number situated at the intersection of the column and the row. Add this number to the number in step (iii). Prefix a decimal point to this number.

For example:

Mantissa of log 0.3745 is (5729 + 6) = 5735 prefixed with the decimal point, i.e., 0.5735

	0	1	2	3	4	...	1 2 3 4 5 ...
...					↓		↓
37					5729 →	→	+ → 6

Note: If the characteristic of the logarithm of a number (which is between 0 and 1) is a negative integer, then we keep the decimal part (mantissa) positive and put the bar (for negative) on the top of the characteristic.

For example:

$\log 0.000865 = -4 + 0.9375$

$= \bar{4}.9375$

Meaning of $\bar{4}.9375$ is $-4 + 0.9375$, not -4.9375.

Example 17: *Use logarithm tables to find the logarithm of the following numbers:*

(i) 0.5675 (ii) 0.002359

(iii) 0.00074 (iv) 0.005

Solution:

(i) Write the given number in standand form as:

$0.5675 = 5.675 \times 10^{-1}$

$\Rightarrow$ Characteristic of log 0.5675 is –1.

To find mantissa, we take the first four digits beginning with first non-zero digit on the right of the decimal point. The digits are 5675. Now, first look in the row starting with 56. In this row, look at the number in the column headed by 7. The number is 7536.

Now, more to the column of mean difference and look under the column headed by 5 in the corresponding to 56. We find the number 4 there. Add this number to 7536. We get the number (7536 + 4) = 7540. This is the required mantissa of log 0.5675.

Thus,

log 0.5675 =Characteristic + Mantissa

= –1 + 0.7540

= 1.7540 (= – 0.2460)

	0	1	2	...	7	...	1 2 3 4 5 ... 9
56					7536		4
...							

(ii) Write the given number in standard form as:

$0.002359 = 2.359 \times 10^{-3}$

⇒ Characteristic of log 0.002359 = Power of 10 standard form

= –3

We take the first four digits beginning with first non-zero digit on the right of the decimal point. The digit are 2359. Using logarithm tables, the mantissa of log 0.002359 is given by:

3711 + 17 = 3728

	0 1 ...	5	...	1 2 3 ... 9
...				
23		3711		17

Thus,

log 0.002359 = –3 + 0.3711 (+17 added in the decimal point)

= –3 + 0.3728

= $\bar{3}.3728$

(iii) Write the given number in standand form as:

$0.00074 = 7.4 \times 10^{-4}$

⇒ Characteristic of log 0.00074 = Power of to is standed form

= –4

To find the mantissa, we take only 74. Look in the row 74 under column 0. We see the number 8692 there.

Therefore, the required mantissa = 8692

$\therefore$ $\log 0.00074 = -4 + 0.8692 = \bar{4}.8692$

	0	1 2 ...	...
	↓		
74	8692		

(iv) Write the given number in standard form as:

$$0.005 = 5 \times 10^{-3}$$

$\Rightarrow$ Characteristic of log 0.005 = –3

To find mantissa, we consider the number 50.

Now, we look in the row 50 under column headed by 0. We see the number 6990.

$\therefore$ $\log (0.005) = -3 + 0.6990$

$= \bar{3}.6990$

	0	1 2 ...	...
	↓		
74	6990		

EXERCISE 2.7

1. Write the characteristic of the logarithms of each of the following numbers by using their standard forms:

(i) 2315.4 (ii) 459.72

(iii) 45.852 (iv) 5.29385

(v) 0.23145 (vi) 0.025138

(vii) 0.001238 (viii) 0.0001052

2. Write the significant digits in each of the following numbers to compute the mantissa of their logarithms:

(i) 2.185 (ii) 7
(iii) 0.5 (iv) 0.05
(v) 0.0512 (vi) 25
(vii) 0.0003 (viii) 0.00031
(ix) 0.00002015

3. Use logarithm tables to find the logarithms of each of the following numbers. Use calculator to verity result.

(i) 12 (ii) 367
(iii) 125.35 (iv) 20.125
(v) 1046 (vi) 52100
(vii) 25795 (viii) 380950

4. Using logarithm tables to find the logarithm of the following:

(i) 0.53498 (ii) 0.02539
(iii) 0.00375 (iv) 0.000125
(v) 0.00003258

2.11. LAWS OF LOGARITHMS

Remembering that a logarithm is an exponent. In this section, we shall take logarithms to any base a ($a > 0$ and $a \neq 1$).

First Law. (Product Rule)

If m and n are positive real numbers, then

$$\log_a (mn) = \log_a m + \log_a n.$$

In words: The logarithm of a product of two numbers is equal to the sum of their logarithms.

For example:

(i) $\log_3 (5 \times 7) = \log_3 5 + \log_3 7$

(ii) $\log_{10} 30 = \log_{10} (10 \times 3) = \log_{10} 10 + \log_{10} 3 = 1 + \log_{10} 3$

(iii) $\log_a (xyz) = \log_a ((xy)z) = \log_a (xy) + \log_a z = \log_a x + \log_a y + \log_{10} z$

Second Law. (Quotient Rule)

If m and n are positive real numbers, then

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

In words: The logarithm of a quotient/ratio of two numbers is equal to the difference of their logarithms.

For example:

(i) $\log_2 \left(\frac{17}{11}\right) = \log_2 17 - \log_2 11$

(ii) $\log_3 \left(\frac{15}{25}\right) = \log_3 \left(\frac{3}{5}\right) = \log_3 3 - \log_3 5 = 1 - \log_3 5$

(iii) $\log_5 \left(\frac{35}{7}\right) = \log_5 5 = 1$

Third Law. (Power Rule)

If m and n are positive real numbers, then

$$\log_a m^n = n \log_a m$$

In words: The logarithm of the power of a number is the product of the power and the logarithm of the number.

For example:

(i) $\log_5 1000 = \log_5 10^3 = 3 \log_5 10 = 3 \log_5 (2 \times 5)$
$= 3 (\log_5 2 + \log_5 5) = 3 (\log_5 2 + 1)$

(ii) $\log_{10} \left(\frac{2^3 \times 5^4}{7^5}\right) = \log_{10} (2^3 \times 5^4) - \log_{10} 7^5$ (By quotient rule)

$= \log_{10} 2^3 + \log_{10} 5^4 - \log_{10} 7^5$ (By product rule)
$= 3 \log_{10} 2 + 4 \log_{10} 5 - 5 \log_{10} 7$ (By power rule)

Notes:

- $\log (mn) \neq \log m \times \log n$
- $\log \left(\frac{m}{n}\right) \neq \frac{\log m}{\log n}$
- $\log (m + n) \neq \log m + \log n$
- $\log (m - n) \neq \log m - \log n$

In fact, there is no sum rule or difference rule.

Example 18: *Using the laws of logarithms, express each of the following as the logarithm of a single number (or only a number):*

(i) $\log 3 + \log 5$ (ii) $\log 24 - \log 8$

(iii) $\log 2 + 2\log 5$ (iv) $\log 4 + \log 25$

(v) $2\log 7 - 3\log 2$ (vi) $2 - 2\log 5$

(vii) $\dfrac{\log 8}{\log 4}$ (viii) $\dfrac{\log 25}{\log 0.2}$

(ix) $\log 3 + \log 8 - \log 4$ (x) $\log(x^2 - 1) - \log(x + 1)$

Solution:

(i) $\log 3 + \log 5 = \log(3 \times 5) = \log 15$

(ii) $\log 24 - \log 8 = \log\left(\dfrac{24}{8}\right) = \log 3$

(iii) $\log 2 + 2\log 5 = \log 2 + \log 5^2 = \log(2 \times 5^2) = \log 50$

(iv) $\log 4 + \log 25 = \log(4 \times 25) = \log 100 = \log 10^2 = 2\log 10 = 2$

(v) $2\log 7 - 3\log 2 = \log(7)^2 - \log(2)^3 = \log 49 - \log 8 = \log\left(\dfrac{49}{8}\right)$

(vi) $2 - 2\log 5 = \log 10^2 - \log 5^2 = \log 100 - \log 25 = \log\left(\dfrac{100}{25}\right) = \log 4$

(vii) $\dfrac{\log 8}{\log 4} = \dfrac{\log 2^3}{\log 2^2} = \dfrac{3\cancel{\log 2}}{2\cancel{\log 2}} = \dfrac{3}{2}$

(viii) $\dfrac{\log 25}{\log 0.2} = \dfrac{\log 5^2}{\log\left(\dfrac{2}{10}\right)} = \dfrac{2\log 5}{\log\left(\dfrac{1}{5}\right)} = \dfrac{2\log 5}{\log 5^{-1}} = \dfrac{2\log 5}{-\log 5} = -2$

(ix) $\log 3 + \log 8 - \log 4 = \log(3 \times 8) - \log 4 = \log 24 - \log 4$

$= \log\left(\dfrac{24}{4}\right) = \log 6$

(x) $\log(x^2 - 1) - \log(x + 1) = \log(x + 1)(x - 1) - \log(x - 1)$

$= \log(x + 1) + \log(x - 1) - \log(x - 1)$

$= \log(x + 1)$

Example 19: *Simpily:*

(i) $\log_3 15 + \log_3 8$

(ii) $4\log_5 4 - 8\log_5 2$

(iii) $\frac{1}{2}\log_7 25 - 2\log_7 3 + 2\log_7 6$

(iv) $2 \log_a 5 + \log_a 4 - 2 \log_a 10$

(v) $\log_9 \frac{75}{16} + \log_9 \frac{32}{243} - 2\log_9 \frac{5}{9}$

Solution:

(i) $\log_3 15 + \log_3 8 = \log_3 (15 \times 8) = \log_3 120$

(ii) $4 \log_5 4 - 8 \log_5 2 = \log_5 4^4 - \log_5 2^8 = \log_5 256 - \log_5 256 = 0$

(iii) $\frac{1}{2}\log_7 25 - 2\log_7 3 + 2\log_7 6 = \log_7 (25)^{\frac{1}{2}} - \log_7 3^2 + \log_7 6^2$

$= \log_7 \left(5^2\right)^{\frac{1}{2}} - \log_7 9 + \log_7 36 = \log_7 5 + \log_7 36 - \log_7 9$

$= \log_7 (5 \times 36) - \log_7 9 = \log_7 \left(\frac{5 \times 36}{9}\right) = \log_7 20$

(iv) $2 \log_4 5 + \log_4 4 - 2 \log_4 10 = \log_4 5^2 + \log_4 4 - \log_4 10^2$
$= \log_4 25 + \log_4 4 - \log_4 100 = \log_4 (25 \times 4) - \log_4 100$
$= \log_4 100 - \log_4 100 = 0$

(v) $\log_9 \frac{75}{16} + \log_9 \frac{32}{243} - 2\log_9 \frac{5}{9}$

$= \log_9 \left(\frac{\cancel{75}^{25}}{\cancel{16}} \times \frac{\cancel{32}^{2}}{\cancel{243}^{81}}\right) - \log_9 \left(\frac{5}{9}\right)^2$

$= \log_9 \left(\frac{50}{81}\right) - \log_9 \frac{25}{81}$

$= \log_9 \left(\frac{\frac{50}{81}}{\frac{25}{81}}\right) = \log_9 \left(\frac{\cancel{50}^{2}}{\cancel{81}} \times \frac{\cancel{81}}{\cancel{25}}\right) = \log_9 2$

Example 20: *Given $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$. Find without using tables, the values of*

(i) $\log_{10} 25$

(ii) $\log_{10} 72$

(iii) $\log_{10} 0.4$

Solution:

(i) $\log_{10} 25 = \log_{10} 5^2 = 2 \log_{10} 5 = 2 \log_{10} \left(\frac{10}{2}\right)$

$= 2[\log_{10} 10 - \log_{10} 2]$

$= 2\,[1 - 0.3010] = 2[0.6990] = 1.3980$

(ii) $\log_{10} 72 = \log_{10} (2 \times 2 \times 2 \times 3 \times 3) = \log_{10} (2^3 \times 3^2)$

$= \log_{10} 2^3 + \log_{10} 3^2 = 3 \log_{10} 2 + 2 \log_{10} 3$

$= 3 \times 0.3010 + 2 \times 0.4771$

$= 0.9030 + 0.9542 = 1.8572$

(iii) $\log_{10} 0.4 = \log_{10} \left(\frac{4}{10}\right) = \log_{10} 4 - \log_{10} 10 = \log_{10} 2^2 - \log_{10} 10$

$= -\log_{10} 10 + 2 \log 2$

$= -1 + 2\,(0.3010) = -1 + (0.6020) = \bar{1}.6020$

Example 21: *Solve for x:*

(i) $\log 2x = 1.$

(ii) $\frac{\log 64}{\log 16} = 3x.$

(iii) $\log (x + 2) + \log (x - 2) = \log 5.$

Solution:

(i) Given $\log 2x = 1$

$\Rightarrow \log 2x = \log 10$ (Since $1 = \log 10$)

$\Rightarrow 2x = 10$

$\Rightarrow x = \frac{10}{2} = 5.$

(ii) Given $\frac{\log 64}{\log 16} = 3x$

$\Rightarrow \frac{\log 2^6}{\log 2^4} = 3x \Rightarrow \frac{6 \log 2}{4 \log 2} = 3x$

$\Rightarrow \frac{6}{4} = 3x \Rightarrow 3x = \frac{3}{2} \quad \Rightarrow \quad x = \frac{1}{2}.$

(iii) Given log $(x + 2)$ + log$(x - 2)$ = log 5

$\Rightarrow$ log$(x + 2)(x - 2)$ = log 5 $\Rightarrow$ log $(x^2 - 4)$ = log 5

$\Rightarrow$ $x^2 - 4 = 5$ $\Rightarrow$ $x^2 = 5 + 4$

$\Rightarrow$ $x^2 = 9$ $\Rightarrow$ $x = 3$.

EXERCISE 2.8

1. Using rules of logarithms, expand each of the following as much as possible:

(i) $\log(15\times25)$

(ii) $\log\frac{100}{81}$

(iii) $\log(5^2\times7^6)$

(iv) $\log\left(\frac{3^2\times7^3}{2^5}\right)$

2. Simplify:

(i) log 3 + log 4 + log 5 – log 6

(ii) log 200 + log 5

(iii) log 5 + log 20 + log 24 + log 25 – log 60

(iv) 2 log 3 + 3 log 4 + 4 log 5 – 2 log 6

3. Express each of the following as the logarithm of a single number:

(i) log 2 + 1

(ii) log 12 – log 2 – log 3

(iii) 3 log 2 + 2

(iv) log $2x$ + 2 log x

(v) log a^2 – log a

(vi) $\frac{1}{2}\log 9+\frac{1}{4}\log 81+2\log 6-\log 12$

(vii) $\frac{1}{2}\log 9-3\log 4+3\log 2$

(viii) $1-\frac{1}{3}\log 64$

4. Evaluate each of the following:

(i) 4 log 5 + 2 log 4

(ii) log 6 + 2 log 5 + log 4 – log 3 – log 2

(iii) $\frac{1}{2}\log 36 + \log 5 - \log 30$

(iv) $\log 5 + 2 \log 0.5 + 3 \log 2$

5. Find the values of x in each of the following:

(i) $\frac{\log 144}{\log 12} = \log x$ (ii) $\frac{\log 125}{\log 25} = x$

(iii) $\log_x 4 + \log_x 16 + \log_x 64 = 12$

6. Solve for x:

(i) $\log x + \log 4 = \log 12$

(ii) $\log x + \log 5 = 1$

(iii) $\log (x + 1) + \log (x - 1) = \log 3$

(iv) $\log(x + 1) - \log(x - 1) = 1$

7. Given $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, find

(i) $\log_{10} 12$ (ii) $\log_{10}\left(\frac{9}{2}\right)$

(iii) $\log_{10} 54$

MULTIPLE CHOICE QUESTIONS

1. If a is any real number and x, y are +ve integers, then, which one of the following is true?

(a) $(a^x)^y = a^{xy}$ (b) $(a^y)^x = a^{xy}$

(c) $(a^y)^x = (a^x)^y$ (d) all of these

2. For any number n, the formula $a^{-n} = \frac{1}{a^n}$ holds good if

(a) $a \neq 0$ (b) $a \neq 1$

(c) $a \neq 7$ (d) a is any real number

3. The solution of the equation over the set of natural numbers is

$$3^{x^2 - 1} = 6561$$

(a) $x = 3$ (b) $x = -3$

(c) $x = \pm 3$ (d) all of these

4. The value of x for which $8^x = 0.25$ holds good over the set of integers is

(a) $\frac{-2}{3}$ (b) -2

(c) -3 (d) none of these

5. The solution of the simultaneous equation $\left(\frac{1}{3}\right)^{2x} = \left(\frac{1}{9}\right)^{y+2}$ is

(a) $x = 3, y = 5$ (b) $x = 5, y = 3$

(c) $x = 3, y = -5$ (d) $x = -3, y = 5$

6. The equation $10^{-1} = 0.1$ when converted to logarithm becomes

(a) $\log 0.1 = -1$ (b) $\log_{10} 0.1 = -1$

(c) $\log_{10} 0.1 = 1$ (d) none of these

7. The logarithm equation $\log_{10} 0.01 = -3$ when converted to exponential form becomes

(a) $10^{-3} = 0.01$ (b) $10^{3} = 0.01$

(c) $10^{-3} = 0.001$ (d) $10^{3} = 0.001$

8. The law of logarithm $\log_a x - \log_a y = \log_a \frac{x}{y}$ holds good if

(a) $x > 0, y > 0$ (b) $x > 0, y < 0$

(c) $x < 0, y < 0$ (d) $x < 0, y < 0$

9. Using $\log_3 5 = 1.4650$, the value of $\log_3 25$ is

(a) 2.39 (b) 3.29

(c) 2.93 (d) 3.92

10. The fraction $\frac{(\log_{10} 125)(\log_{10} 36)}{(\log_{10} 216)(\log_{10} 625)}$ when simplified is equivalent to

(a) 2 (b) $\frac{1}{2}$

(c) 1 (d) none of these

11. The value of $\log_{10} 6 + \log_{10} 45 - \log_{10} 27$ is

(a) 0 (b) 1

(c) 2 (d) 3

12. The number of solutions of the equation $\log_{10} 2x + \log_{10} (x - 4) = 1$ is

(a) 1 (b) 2

(c) 3 (d) 4

13. The characteristic of log 567 is

(a) 0 (b) 3

(c) 1 (d) 2

14. The mantissa of log 0.0345 is

(a) log 3.45 (b) 0.5378

(c) both (a) and (b) (d) none of these